NSW Education Standards Authority

2019 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–14)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the domain of the function $f(x) = \ln(4 - x)$?

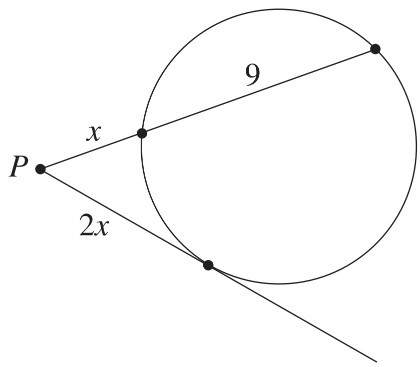
A. $x < 4$

B. $x \leq 4$

C. $x > 4$

D. $x \geq 4$

2 From the point P outside a circle, a secant and a tangent to the circle are constructed as shown in the diagram.



NOT TO SCALE

Which equation is satisfied by x?

A. $4x^2 = 9x$

B. $4x^2 = 9 + x$

C. $4x^2 = 9(9 + x)$

D. $4x^2 = x(9 + x)$

3 What is the derivative of $\tan^{-1}\dfrac{x}{2}$?

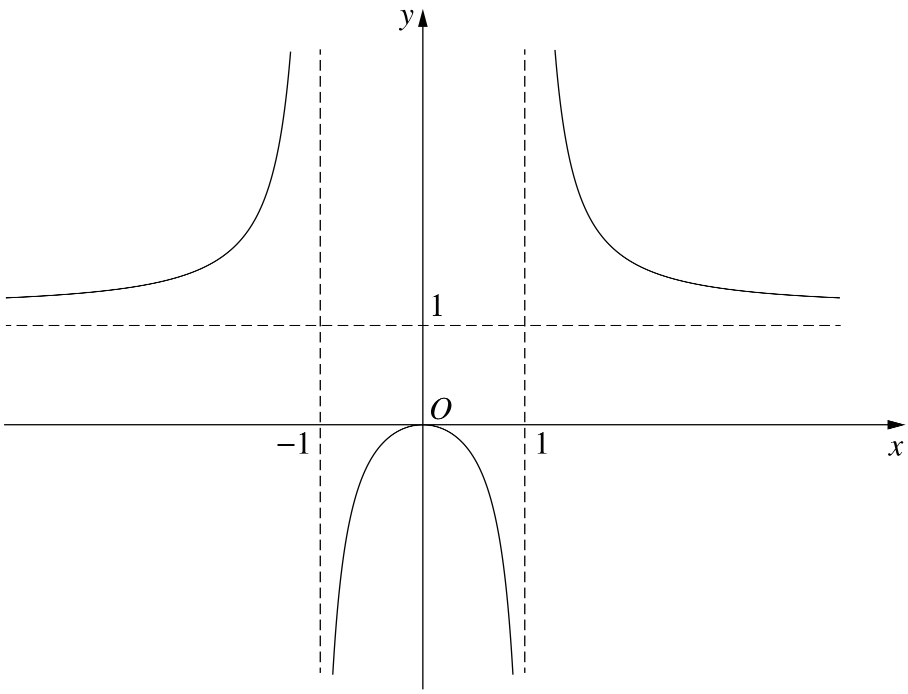
A. $\dfrac{1}{2\left(4+x^2\right)}$

B. $\dfrac{1}{4+x^2}$

C. $\dfrac{2}{4+x^2}$

D. $\dfrac{4}{4+x^2}$

4 The diagram shows the graph of $y = f(x)$.



Which equation best describes the graph?

A. $y = \dfrac{x}{x^2 - 1}$

B. $y = \dfrac{x^2}{x^2 - 1}$

C. $y = \dfrac{x}{1 - x^2}$

D. $y = \dfrac{x^2}{1 - x^2}$

5 A particle starts from rest, 2 metres to the right of the origin, and moves along the x-axis in simple harmonic motion with a period of 2 seconds.

Which equation could represent the motion of the particle?

A. $x = 2\cos \pi t$

B. $x = 2\cos 2t$

C. $x = 2 + 2\sin \pi t$

D. $x = 2 + 2\sin 2t$

6 It is given that $\sin x = \frac{1}{4}$, where $\frac{\pi}{2} < x < \pi$.

What is the value of $\sin 2x$?

A. $-\frac{7}{8}$

B. $-\frac{\sqrt{15}}{8}$

C. $\frac{\sqrt{15}}{8}$

D. $\frac{7}{8}$

7 Let $P(x) = qx^3 + rx^2 + rx + q$ where q and r are constants, $q \neq 0$. One of the zeros of $P(x)$ is -1.

Given that α is a zero of $P(x)$, $\alpha \neq -1$, which of the following is also a zero?

A. $-\frac{1}{\alpha}$

B. $-\frac{q}{\alpha}$

C. $\frac{1}{\alpha}$

D. $\frac{q}{\alpha}$

8 In how many ways can all the letters of the word PARALLEL be placed in a line with the three Ls together?

A. $\dfrac{6!}{2!}$

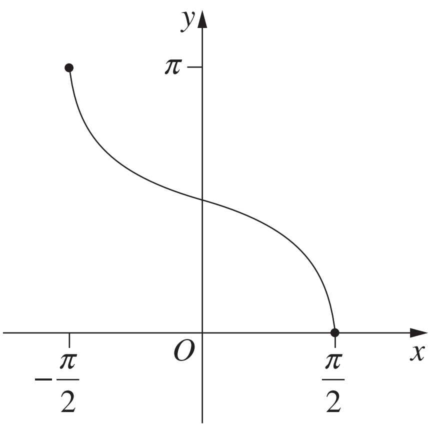
B. $\dfrac{6!}{2!3!}$

C. $\dfrac{8!}{2!}$

D. $\dfrac{8!}{2!3!}$

9 Which graph best represents $y = \cos^{-1}(-\sin x)$, for $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$?

A.



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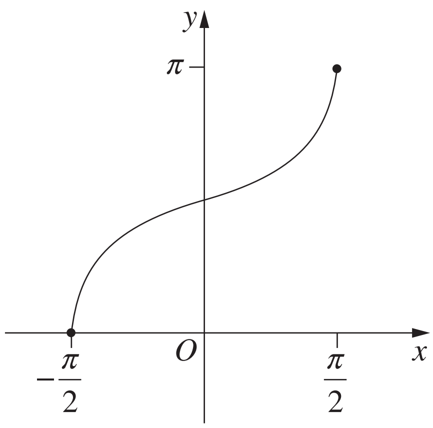

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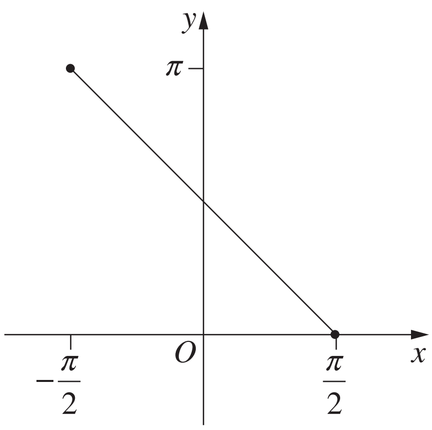
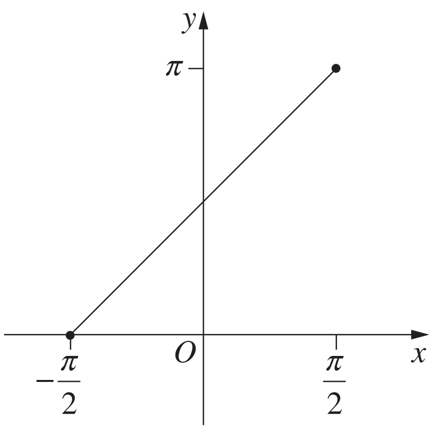


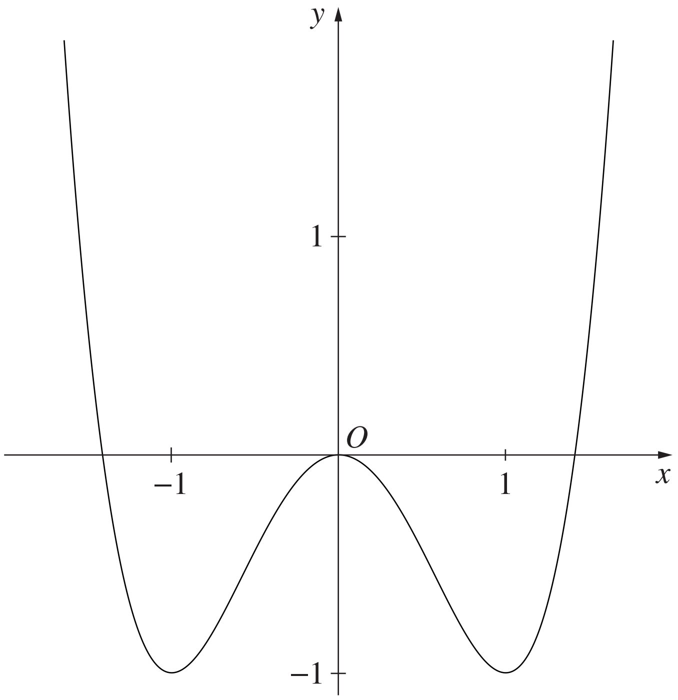
D.



10 The function $f(x) = -\sqrt{1+\sqrt{1+x}}$ has inverse $f^{-1}(x)$.

The graph of $y = f^{-1}(x)$ forms part of the curve $y = x^4 - 2x^2$.

The diagram shows the curve $y = x^4 - 2x^2$.



How many points do the graphs of $y = f(x)$ and $y = f^{-1}(x)$ have in common?

A. 1

B. 2

C. 3

D. 4

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

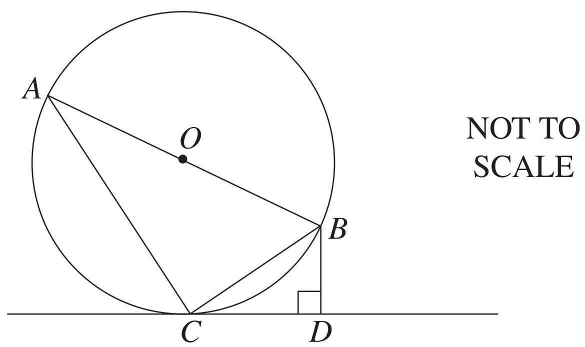
In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Find the acute angle between the lines $x - 2y + 1 = 0$ and $y = 3x - 4$. **2**

(b) For what values of x is $\dfrac{x}{x+1} < 2$? **3**

(c) The line segment AB is a diameter of the circle, centred at O. The line CD is the tangent to the circle at C and BD is perpendicular to CD, as shown in the diagram. **3**



Prove that $\angle ABC = \angle CBD$.

(d) Find the polynomial $Q(x)$ that satisfies $x^3 + 2x^2 - 3x - 7 = (x - 2)Q(x) + 3$. **2**

Question 11 continues on page 8

(e) Find $\displaystyle\int 2\sin^2 4x\,dx$. **2**

(f) Prize-winning symbols are printed on 5% of ice-cream sticks. The ice-creams are randomly packed into boxes of 8.

(i) What is the probability that a box contains no prize-winning symbols? **1**

(ii) What is the probability that a box contains at least 2 prize-winning symbols? **2**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Distance A is inversely proportional to distance B, such that $A = \dfrac{9}{B}$, where A and B are measured in metres. The two distances vary with respect to time. Distance B is increasing at a rate of 0.2 m s^{-1}. **3**

What is the value of $\dfrac{dA}{dt}$ when $A = 12$?

(b) A particle is moving along the x-axis in simple harmonic motion. The position of the particle is given by

$$x = \sqrt{2}\cos 3t + \sqrt{6}\sin 3t, \text{ for } t \geq 0.$$

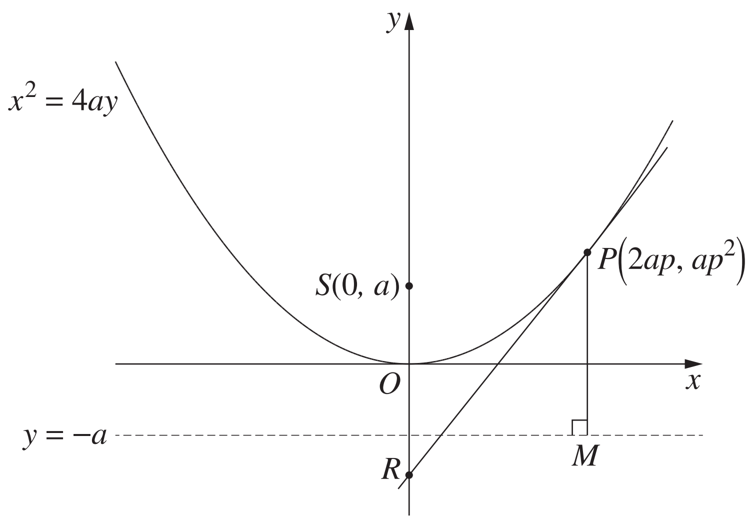
(i) Write x in the form $R\cos(3t - \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$. **2**

(ii) Find the two values for x where the particle comes to rest. **1**

(iii) When is the first time that the speed of the particle is equal to half of its maximum speed? **2**

Question 12 continues on page 10

(c) The point $P\left(2ap, ap^2\right)$ lies on the parabola $x^2 = 4ay$. The focus of the parabola is at $S(0, a)$ and the directrix is the line $y = -a$. The vertical line through P meets the directrix at M. The tangent at P meets the y-axis at R. **3**



Show that $\angle SPR = \angle SRP$.

(d) A refrigerator has a constant temperature of 3°C. A can of drink with temperature 30°C is placed in the refrigerator.

After being in the refrigerator for 15 minutes, the temperature of the can of drink is 28°C.

The change in the temperature of the can of drink can be modelled by $\dfrac{dT}{dt} = k(T - 3)$, where T is the temperature of the can of drink, t is the time in minutes after the can is placed in the refrigerator and k is a constant.

(i) Show that $T = 3 + Ae^{kt}$, where A is a constant, satisfies $\dfrac{dT}{dt} = k(T - 3)$. **1**

(ii) After 60 minutes, at what rate is the temperature of the can of drink changing? **3**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) Use the substitution $u = \cos^2 x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{4 + \cos^2 x}\,dx$. **3**

(b) In the expansion of $(5x + 2)^{20}$, the coefficients of x^k and x^{k+1} are equal. **3**

What is the value of k?

(c) A particle moves in a straight line. At time t seconds the particle has a displacement of x m, a velocity of v m s^{-1} and acceleration a m s^{-2}. Initially the particle has displacement 0 m and velocity 2 m s^{-1}. The acceleration is given by $a = -2e^{-x}$. The velocity of the particle is always positive.

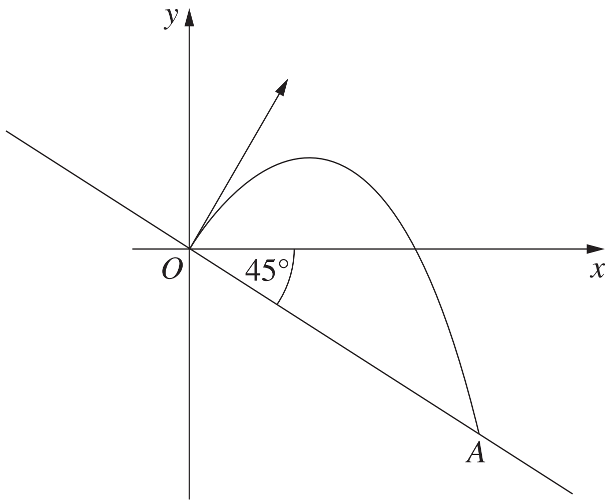
(i) Show that $v = 2e^{\frac{-x}{2}}$. **2**

(ii) Find an expression for x as a function of t. **2**

Question 13 continues on page 12

Question 13 (continued)

(d) The point O is on a sloping plane that forms an angle of 45° to the horizontal. A particle is projected from the point O. The particle hits a point A on the sloping plane as shown in the diagram.



The equation of the line OA is $y = -x$. The equations of motion of the particle are

$$x = 18t$$
$$y = 18\sqrt{3}t - 5t^2,$$

where t is the time in seconds after projection. Do NOT prove these equations.

(i) Find the distance OA between the point of projection and the point where the particle hits the sloping plane. **2**

(ii) What is the size of the acute angle that the path of the particle makes with the sloping plane as the particle hits the point A? **3**

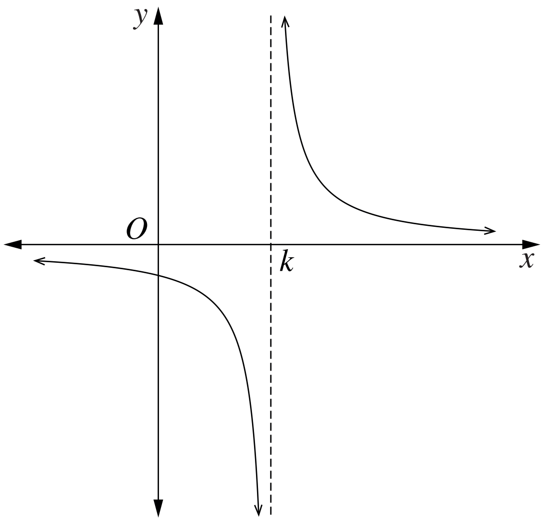
End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) Prove by mathematical induction that, for all integers $n \geq 1$, **3**

$$1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1.$$

(b) The diagram shows the graph of $y = \dfrac{1}{x-k}$, where k is a positive real number.



(i) By considering the graphs of $y = x^2$ and $y = \dfrac{1}{x-k}$, explain why the function $f(x) = x^3 - kx^2 - 1$ has exactly one real zero. **2**

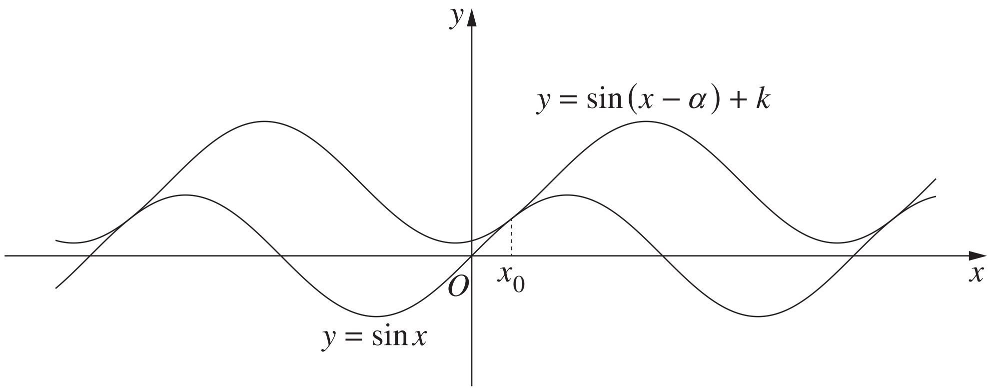
Let this zero be α and let $x_1 = k$ be a first approximation to α.

(ii) Show that using one application of Newton's method gives a second approximation, $x_2 = k + \dfrac{1}{k^2}$. **2**

(iii) Show that $x_1 < \alpha < x_2$. **3**

Question 14 continues on page 14

(c) The diagram shows the two curves $y = \sin x$ and $y = \sin(x - \alpha) + k$, where $0 < \alpha < \pi$ and $k > 0$. The two curves have a common tangent at x_0 where $0 < x_0 < \dfrac{\pi}{2}$.



(i) Explain why $\cos x_0 = \cos(x_0 - \alpha)$. **1**

(ii) Show that $\sin x_0 = -\sin(x_0 - \alpha)$. **2**

(iii) Hence, or otherwise, find k in terms of α. **2**

End of paper

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2019 HIGHER SCHOOL CERTIFICATE EXAMINATION

REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

Angle sum of a polygon

$$S = (n-2) \times 180^\circ$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

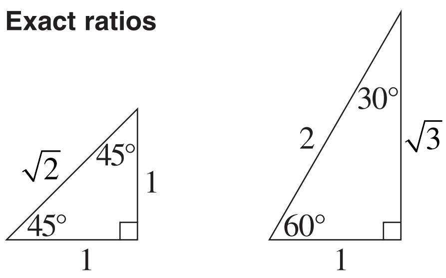
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{\left|ax_1 + by_1 + c\right|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point–gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a\left(r^n - 1\right)}{r-1} \quad \text{or} \quad S_n = \frac{a\left(1 - r^n\right)}{1-r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1-r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Derivatives

If $y=x^n$, then $\dfrac{dy}{dx}=nx^{n-1}$

If $y=uv$, then $\dfrac{dy}{dx}=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$

If $y=\dfrac{u}{v}$, then $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

If $y=F(u)$, then $\dfrac{dy}{dx}=F'(u)\dfrac{du}{dx}$

If $y=e^{f(x)}$, then $\dfrac{dy}{dx}=f'(x)e^{f(x)}$

If $y=\log_e f(x)=\ln f(x)$, then $\dfrac{dy}{dx}=\dfrac{f'(x)}{f(x)}$

If $y=\sin f(x)$, then $\dfrac{dy}{dx}=f'(x)\cos f(x)$

If $y=\cos f(x)$, then $\dfrac{dy}{dx}=-f'(x)\sin f(x)$

If $y=\tan f(x)$, then $\dfrac{dy}{dx}=f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$$\alpha+\beta=-\frac{b}{a}\qquad \alpha\beta=\frac{c}{a}$$

Equation of a parabola

$$(x-h)^2=\pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+C$$

$$\int e^{ax+b}\,dx=\frac{1}{a}e^{ax+b}+C$$

$$\int \frac{f'(x)}{f(x)}\,dx=\ln\left|f(x)\right|+C$$

$$\int \sin(ax+b)\,dx=-\frac{1}{a}\cos(ax+b)+C$$

$$\int \cos(ax+b)\,dx=\frac{1}{a}\sin(ax+b)+C$$

$$\int \sec^2(ax+b)\,dx=\frac{1}{a}\tan(ax+b)+C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx\approx\frac{b-a}{2}\left[f(a)+f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx\approx\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

Logarithms – change of base

$$\log_a x=\frac{\log_b x}{\log_b a}$$

Angle measure

$180°=\pi$ radians

Length of an arc

$l=r\theta$

Area of a sector

$\text{Area}=\dfrac{1}{2}r^2\theta$

Mathematics Extension 1

Angle sum identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

***t* formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a, \qquad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \qquad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \qquad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2 + nx_1}{m+n},\ \frac{my_2 + ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $\left(2at,\ at^2\right)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At $\left(x_1,\ y_1\right)$,

tangent: $xx_1 = 2a\left(y + y_1\right)$

normal: $y - y_1 = -\frac{2a}{x_1}\left(x - x_1\right)$

Chord of contact from $\left(x_0,\ y_0\right)$: $xx_0 = 2a\left(y + y_0\right)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$$x = b + a\cos\left(nt + \alpha\right)$$

$$\ddot{x} = -n^2\left(x - b\right)$$

Further integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f\left(x_1\right)}{f'\left(x_1\right)}$$

Binomial theorem

$$\left(a+b\right)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$